Low-Rank Univariate Sum of Squares Has No Spurious Local Minima

ICCOPT 2022

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MIT

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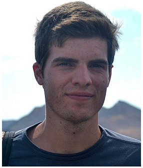

Introduction

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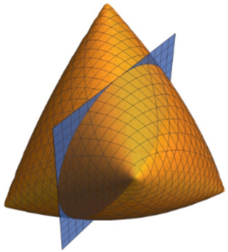
Positive semidefinite variable $X \succeq 0$ + linear constraints

Solved in polynomial time with interior point methods ($n \sim 10^3$)



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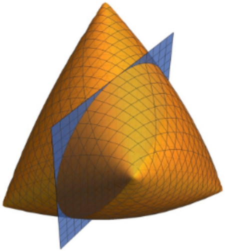

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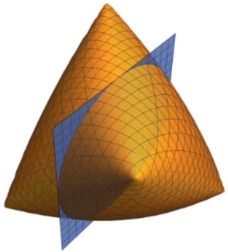

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Can we apply these ideas to solving SDPs?

The Burer-Monteiro Method

Burer-Monteiro methods for solving SDPs factor PSD variable $X = UU^T$, then perform local optimization on *non-convex* unconstrained problem

$$\begin{array}{rl} \langle A_i, X\rangle & = b_i \quad \forall i \\ X & \succeq 0 \end{array} \quad \longrightarrow \quad \min_U \sum_i (\langle A_i, UU^T\rangle - b_i)^2$$

$$\text{Feasible} \quad \Longleftrightarrow \quad \text{Optimum} = 0$$

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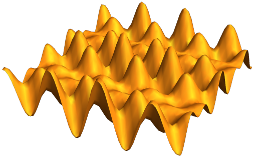

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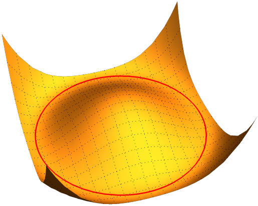

When is non-convexity benign?

Related Work

SDP with m linear constraints, factorization $X = UU^\top$, where $U \in \mathbb{R}^{n \times r}$.

[BM05] Burer and Monteiro. "Local Minima and Convergence in Low-Rank Semidefinite Programming". 2005.

[CM19] Cifuentes and Moitra. "Polynomial Time Guarantees for the Burer-Monteiro Method". 2019.

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Second-order critical points $\implies$ **Global minima** (non-convexity benign):

- $r \geq n$ [BM05] (explicit counterexamples exist for $r = n - 1$, $m = n$)
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Smaller r in factorization $\rightarrow$ less benign landscape

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Can we get do better if the SDP has special structure?

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Sum of Squares Optimization

Given $p(x)$, can we write it as a sum of squares? $p(x) = \sum_{i=1}^{r} u_i(x)^2$

Certifies that $p(x) \geq 0$, and can be formulated as a SDP:

$$p(x) = \vec{b}(x)^\top Q \vec{b}(x), \quad Q \succeq 0$$

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Q satisfying above constraints is called the Gram spectrahedron [Chu+16]

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Previous work: rank needed for benign non-convexity $\sim$ max rank of extreme points of Gram spectrahedron

Can we do better?

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Image credit: Tae Roh and Lieven Vandenberghe. (2006) Discrete transforms, semidefinite programming and sum-of-squares representations of nonnegative polynomials. SIAM J. on Optimization.

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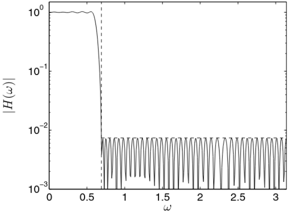
Univariate (trigonometric) polynomials:

$$p(x) = a_0 + \sum_{k=1}^{2d} a_k \cos(kx) \quad x \in [0, 2\pi]$$



Applications in signal processing, filter design and control

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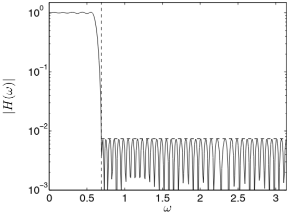

Applications in signal processing, filter design and control

Gram spectrahedra has extreme points of all ranks: $2 \leq r \lesssim \sqrt{d}$ [Sch22]

But always has rank-2 point! (Sum of 2 squares)

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Contributions

Find sum of squares decomposition of $p(x)$ by solving (equivalent to B-M):

$$\min_{\mathbf{u}} f_p(\mathbf{u}) = \left\| p(x) - \sum_{i=1}^r u_i(x)^2 \right\|^2$$

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Theorem

For all nonnegative univariate polynomials $p(x) \in \mathbb{R}[x]_{2d}$ and any $r \geq 2$, if $\mathbf{u} \in \mathbb{R}[x]_d^r$ satisfies $\nabla f_p(\mathbf{u}) = 0$ and $\nabla^2 f_p(\mathbf{u}) \succeq 0$, then $f_p(\mathbf{u}) = 0$.

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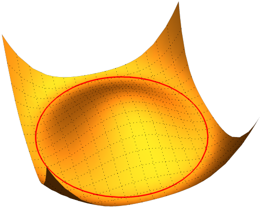
First-order methods find sum of squares decomposition (non-convexity benign)

If we choose a suitable norm, $\nabla f_p(\mathbf{u})$ can be computed in $O(d \log d)$ time using Fast Fourier Transforms (FFTs)



Proof Sketch

Define Sylvester map $\mathcal{A}_{\mathbf{u}} : \mathbb{R}[x]_d^r \to \mathbb{R}[x]_{2d}$

$$\mathcal{A}_{\mathbf{u}}(\mathbf{v}) = \mathcal{A}_{(u_1,u_2)}((v_1, v_2)) = u_1 v_1 + u_2 v_2$$

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$$\begin{aligned} f_p(\mathbf{u}) &= \left\| u_1^2 + u_2^2 - p \right\|^2 \\ \nabla f_p(\mathbf{u})(\mathbf{v}) &\sim \left\langle \mathcal{A}_{\mathbf{u}}(\mathbf{v}), u_1^2 + u_2^2 - p \right\rangle \\ \nabla^2 f_p(\mathbf{u})(\mathbf{v}, \mathbf{v}) &\sim \left\langle \mathcal{A}_{\mathbf{v}}(\mathbf{v}), u_1^2 + u_2^2 - p \right\rangle + \|\mathcal{A}_{\mathbf{u}}(\mathbf{v})\|^2 \end{aligned}$$

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To do so, for every $p \in \Sigma[x]_{2d}$ and $\mathbf{u} \in \mathbb{R}[x]_d^2$, find $\mathbf{v_i} \in \mathbb{R}[x]_d^2$ so that:

$$\nabla f_p(\mathbf{u})(\mathbf{v_0}) + \sum_{i=1}^{k} \nabla^2 f_p(\mathbf{u})(\mathbf{v_i}, \mathbf{v_i}) = -\left\| u_1^2 + u_2^2 - p \right\|^2 = -f_p(\mathbf{u})$$

Geometric Interpretation

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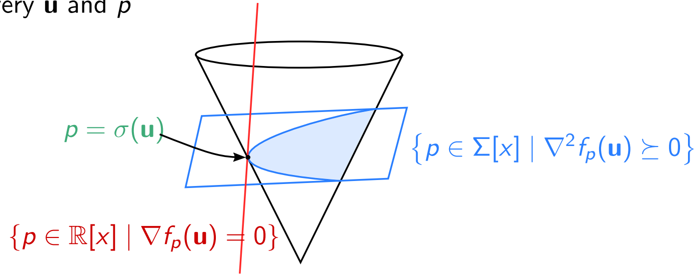

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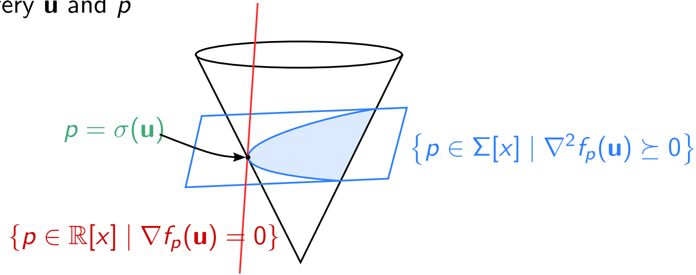

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For fixed $\mathbf{u}$, these sets are convex (and can be represented by SDPs)!

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Suppose $u_1 = u_2$. If $p(x) = 2\sum_i s_i(x)^2$, choose $\mathbf{v_i} = (s_i, -s_i)$, $\mathbf{v_0} = (-u_1, -u_2)$:

$$\nabla f_p(\mathbf{u})(\mathbf{v_0}) = -\left\langle u_1^2 + u_2^2, u_1^2 + u_2^2 - p \right\rangle$$

$$\sum_{i=1}^{k} \nabla^2 f_p(\mathbf{u})(\mathbf{v_i}, \mathbf{v_i}) = \left\langle p, u_1^2 + u_2^2 - p \right\rangle$$

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$$\nabla f_p(\mathbf{u})(\mathbf{v}) \sim \left\langle \mathcal{A}_{\mathbf{u}}(\mathbf{v}), u_1^2 + u_2^2 - p \right\rangle = 0$$

$$\nabla^2 f_p(\mathbf{u})(\mathbf{v}, \mathbf{v}) \sim \left\langle \mathcal{A}_{\mathbf{v}}(\mathbf{v}), u_1^2 + u_2^2 - p \right\rangle + \left\| \mathcal{A}_{\mathbf{u}}(\mathbf{v}) \right\|^2 \geq 0$$

$$-\left\| u_1^2 + u_2^2 - p \right\|^2 = \nabla f_p(\mathbf{u})(\mathbf{v_0}) + \sum_{i=1}^{k} \nabla^2 f_p(\mathbf{u})(\mathbf{v_i}, \mathbf{v_i})$$

Suppose u_1, u_2 are coprime (true generically)

Bézout's lemma ($\mathcal{A}_{\mathbf{u}}$ is onto) $\implies$ there exist $\mathbf{v_0}$ such that

$$\mathcal{A}_{\mathbf{u}}(\mathbf{v_0}) = -(u_1^2 + u_2^2 - p) \implies \nabla f_p(\mathbf{u})(\mathbf{v_0}) = -\left\| u_1^2 + u_2^2 - p \right\|^2$$

Suppose $u_1 = u_2$. If $p(x) = 2 \sum_i s_i(x)^2$, choose $\mathbf{v_i} = (s_i, -s_i)$, $\mathbf{v_0} = (-u_1, -u_2)$:

$$\nabla f_p(\mathbf{u})(\mathbf{v_0}) = -\left\langle u_1^2 + u_2^2, u_1^2 + u_2^2 - p \right\rangle$$

$$\sum_{i=1}^{k} \nabla^2 f_p(\mathbf{u})(\mathbf{v_i}, \mathbf{v_i}) = \left\langle p, u_1^2 + u_2^2 - p \right\rangle$$

Main technical result: how to interpolate between these two cases

Sampled Basis

Theorem holds for any inner product $\langle p(x), q(x) \rangle$ on polynomials, which should we choose?

[LP04] Lofberg and Parrilo. "From Coefficients to Samples: A New Approach to SOS Optimization". 2004.
[CP17] Cifuentes and Parrilo. "Sampling Algebraic Varieties for Sum of Squares Programs". 2017.

Sampled Basis

Theorem holds for any inner product $\langle p(x), q(x)\rangle$ on polynomials, which should we choose?

Given $p(x), q(x)$ of degree d, choose $d+1$ points x_k

$$\langle p(x), q(x)\rangle = \sum_{k=1}^{d+1} p(x_k)q(x_k), \quad \|p(x)\|^2 = \sum_{k=1}^{d+1} p(x_k)^2$$

Valid inner product: when x_k distinct, if $\|p(x)\|^2 = 0$ then $p(x) = 0$.

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Sum of squares using a sampled/interpolation basis studied by [LP04] and [CP17].

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How should we choose x_k?

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Numerical Implementation

Compute sum of squares decomposition of degree $4n$ trigonometric polynomial

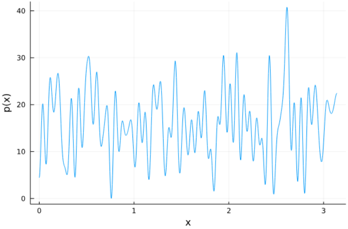
$$p(x) = a_0 + \sum_{k=1}^{2d} a_k \cos(kx) \quad x \in [0, \pi]$$

Using basis vectors evaluated at $4d + 1$ points

$$B_k = [1, \cos(x_k), \ldots, \cos(dx_k)]$$
$$x_k = \frac{k\pi}{d}, \quad k = 1, \ldots, 4d + 1$$

Matrix-vector producted in $\nabla f_p(U)$ computed by FFT

$$\nabla f_p(U) = U^\top B \,\mathrm{Diag}(\left\| U^\top B_k \right\|^2 - p(x_k)) B^\top$$



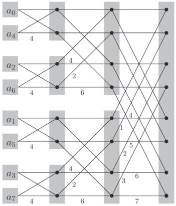
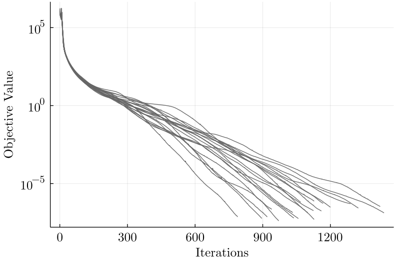


Image credit: Christos Papadimitriou, Sanjoy Dasgupta, and Umesh Vazirani. (2006) Algorithms

Numerical Results

Compute sum of squares decomposition for random trigonometric polynomial

Convergence rate for L-BFGS with random initialization



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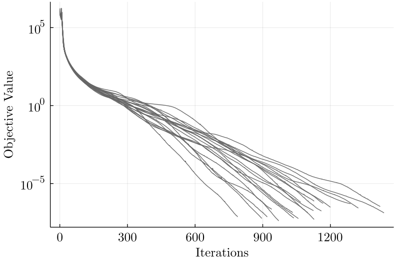

Results (stop at 10^{-7} relative error in $\mathbf{u}$):

Degree of $p(x)$	10,000	20,000	100,000	200,000	1,000,000
Time (s)	6	9	53	160	1461
Iterations	530	632	1126	1375	2303

Conclusion

When does it make sense to solve non-convex formulations of convex problems?

In our setting we can prove that non-convexity does not hurt us

Near-linear time iteration cost with first-order methods in a benign landscape