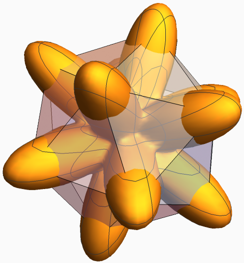
Semidefinite Relaxations of Product of PSD Forms

LIDS Student Conference 2021

Chenyang Yuan (joint work with Pablo Parrilo)

February 3, 2021



How to exploit product structure in polynomial optimization problems

Introduction

How to exploit product structure in polynomial optimization problems

⇓

Computational tractability

How to exploit product structure in polynomial optimization problems

⇓

Computational tractability

+

Provable approximation guarantees

Introduction

Given $\mathcal{A} = (A_1, \dots, A_d)$ where $A_i \succeq 0$, we study the following polynomial optimization problem on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$:

$$\textsc{Opt}(\mathcal{A}) := \max_{x \in \mathbb{K},\, \|x\|=1} \left(\prod_{i=1}^{d} \langle x, A_i x \rangle \right)^{1/d}$$

Introduction

Given $\mathcal{A} = (A_1, \ldots, A_d)$ where $A_i \succeq 0$, we study the following polynomial optimization problem on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$:

$$\textsc{Opt}(\mathcal{A}) \coloneqq \max_{x \in \mathbb{K},\ \|x\|=1} \left(\prod_{i=1}^{d} \langle x, A_i x \rangle \right)^{1/d}$$

Properties:

- High-degree polynomial optimization problem: degree $2d$ in n variables

Introduction

Given $\mathcal{A} = (A_1, \ldots, A_d)$ where $A_i \succeq 0$, we study the following polynomial optimization problem on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$:

$$\textsc{Opt}(\mathcal{A}) := \max_{x \in \mathbb{K},\ \|x\|=1} \left(\prod_{i=1}^{d} \langle x, A_i x \rangle \right)^{1/d}$$

Properties:

- High-degree polynomial optimization problem: degree $2d$ in n variables
- Compact representation: represented in $O(n^2 d)$ space

Introduction

Given $\mathcal{A} = (A_1, \ldots, A_d)$ where $A_i \succeq 0$, we study the following polynomial optimization problem on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$:

$$\textsc{Opt}(\mathcal{A}) \coloneqq \max_{x \in \mathbb{K},\ \|x\|=1} \left(\prod_{i=1}^{d} \langle x, A_i x \rangle \right)^{1/d}$$

Properties:

- High-degree polynomial optimization problem: degree $2d$ in n variables
- Compact representation: represented in $O(n^2 d)$ space
- Polynomial time solution when d is fixed, NP-hard when $d = \Omega(n)$

Applications and Motivation

$$\textsc{Opt}(\mathcal{A}) := \max_{\|x\|=1} \prod_{i=1}^{d} \langle x, A_i x \rangle^{1/d}$$

Applications and Motivation

$$\textsc{Opt}(\mathcal{A}) := \max_{\|x\|=1} \prod_{i=1}^{d} \langle x, A_i x\rangle^{1/d}$$

Kantorovich's inequality ($A_1 = Q$, $A_2 = Q^{-1}$): Given $Q \succ 0$,

$$\max_{\|x\|=1} \sqrt{(x^T A x)(x^T A^{-1} x)} \leq \frac{1}{4}\left(\sqrt{\frac{\lambda_1}{\lambda_n}} + \sqrt{\frac{\lambda_n}{\lambda_1}}\right)^2$$

Applications and Motivation

$$\textsc{Opt}(\mathcal{A}) \coloneqq \max_{\|x\|=1} \prod_{i=1}^{d} \langle x, A_i x \rangle^{1/d}$$

Kantorovich's inequality ($A_1 = Q$, $A_2 = Q^{-1}$): Given $Q \succ 0$,

$$\max_{\|x\|=1} \sqrt{(x^T A x)(x^T A^{-1} x)} \leq \frac{1}{4} \left(\sqrt{\frac{\lambda_1}{\lambda_n}} + \sqrt{\frac{\lambda_n}{\lambda_1}} \right)^2$$

Approximating permanents of PSD matrices [YP20] ($A_i = v_i v_i^\dagger$): Let $M = V^\dagger V$, v_i columns of V.

$$r(M) \coloneqq \max_{\|x\|=1, x \in \mathbb{C}^n} \prod_{i=1}^{n} |\langle x, v_i \rangle|^2, \quad \frac{n!}{n^n} r(M) \leq \mathrm{per}(M)$$

Applications and Motivation

$$\textsc{Opt}(\mathcal{A}) \coloneqq \max_{\|x\|=1} \prod_{i=1}^{d} \langle x, A_i x\rangle^{1/d}$$

Kantorovich's inequality ($A_1 = Q$, $A_2 = Q^{-1}$): Given $Q \succ 0$,

$$\max_{\|x\|=1} \sqrt{(x^T A x)(x^T A^{-1} x)} \leq \frac{1}{4}\left(\sqrt{\frac{\lambda_1}{\lambda_n}} + \sqrt{\frac{\lambda_n}{\lambda_1}}\right)^2$$

Approximating permanents of PSD matrices [YP20] ($A_i = v_i v_i^\dagger$): Let $M = V^\dagger V$, v_i columns of V.

$$r(M) \coloneqq \max_{\|x\|=1, x\in\mathbb{C}^n} \prod_{i=1}^{n} \left|\langle x, v_i\rangle\right|^2, \quad \frac{n!}{n^n} r(M) \leq \mathrm{per}(M)$$

Portfolio optimization ($A_i = \mathrm{diag}(r_i)$): Given rates of return over a time period $r_1, \ldots, r_T \in \mathbb{R}_+^n$, maximize expected profit:

$$\max_{y\geq 0, \sum_i y_i = 1} \left(\prod_{i=1}^{T} \langle y, r_i\rangle\right)^{1/d}$$

Applications and Motivation

$$\textsc{Opt}(\mathcal{A}) \coloneqq \max_{\|x\|=1} \prod_{i=1}^{d} \langle x, A_i x\rangle^{1/d}$$

Kantorovich's inequality ($A_1 = Q$, $A_2 = Q^{-1}$): Given $Q \succ 0$,

$$\max_{\|x\|=1} \sqrt{(x^T A x)(x^T A^{-1} x)} \leq \frac{1}{4}\left(\sqrt{\frac{\lambda_1}{\lambda_n}} + \sqrt{\frac{\lambda_n}{\lambda_1}}\right)^2$$

Approximating permanents of PSD matrices [YP20] ($A_i = v_i v_i^\dagger$): Let $M = V^\dagger V$, v_i columns of V.

$$r(M) \coloneqq \max_{\|x\|=1, x\in\mathbb{C}^n} \prod_{i=1}^{n} \left|\langle x, v_i\rangle\right|^2, \quad \frac{n!}{n^n} r(M) \leq \mathrm{per}(M)$$

Portfolio optimization ($A_i = \mathrm{diag}(r_i)$): Given rates of return over a time period $r_1, \dots, r_T \in \mathbb{R}_+^n$, maximize expected profit:

$$\max_{y\geq 0, \sum_i y_i = 1} \left(\prod_{i=1}^{T} \langle y, r_i\rangle\right)^{1/d}$$

And more! (Solving systems of quadratic equations, linear polarization constants, Nash social welfare ...)

Our Contributions

Using semidefinite programming (SDP) based approximation algorithms for general polynomial optimization (Sum-of-Squares):

Compute: SDP with $n^{O(d)}$ vars/consts Approx: $\Omega(\frac{1}{n})$

Our Contributions

Using semidefinite programming (SDP) based approximation algorithms for general polynomial optimization (Sum-of-Squares):

Compute: SDP with $n^{O(d)}$ vars/consts   Approx: $\Omega(\frac{1}{n})$

Our relaxation and approximation algorithm that exploits compact representation:

Compute: SDP with $n^2 d$ vars/consts   Approx: $\Omega(1)$

Our Contributions

Using semidefinite programming (SDP) based approximation algorithms for general polynomial optimization (Sum-of-Squares):

Compute: SDP with $n^{O(d)}$ vars/consts Approx: $\Omega(\frac{1}{n})$

Our relaxation and approximation algorithm that exploits compact representation:

Compute: SDP with $n^2 d$ vars/consts Approx: $\Omega(1)$

We also:

- Prove that when $d = \Omega(n)$, NP-hard to approximate

Our Contributions

Using semidefinite programming (SDP) based approximation algorithms for general polynomial optimization (Sum-of-Squares):

Compute: SDP with $n^{O(d)}$ vars/consts Approx: $\Omega(\frac{1}{n})$

Our relaxation and approximation algorithm that exploits compact representation:

Compute: SDP with $n^2 d$ vars/consts Approx: $\Omega(1)$

We also:

- Prove that when $d = \Omega(n)$, NP-hard to approximate
- Introduce higher-degree relaxations that trade off computation with approximation quality

Our Contributions

Using semidefinite programming (SDP) based approximation algorithms for general polynomial optimization (Sum-of-Squares):

Compute: SDP with $n^{O(d)}$ vars/consts Approx: $\Omega(\frac{1}{n})$

Our relaxation and approximation algorithm that exploits compact representation:

Compute: SDP with $n^2 d$ vars/consts Approx: $\Omega(1)$

We also:

- Prove that when $d = \Omega(n)$, NP-hard to approximate
- Introduce higher-degree relaxations that trade off computation with approximation quality
- Exhibit integrality gap instances that show our analysis of our SDP based relaxation is tight

Semidefinite Relaxation

$$\max_{\|x\|=1} \prod_{i=1}^{d} \langle x, A_i x \rangle^{1/d} \qquad \text{(OPT)}$$

rank-1 relax ↓   use AM-GM ↓

$$\begin{aligned} \max_{X}\ & \prod_{i=1}^{d} \langle X, A_i \rangle^{1/d} \\ \text{s.t.}\ & \operatorname{Tr}(X) = 1,\ X \succeq 0 \end{aligned} \quad \overset{\text{dual}}{\longleftrightarrow} \quad \begin{aligned} \min_{\lambda,\alpha}\ & \lambda \\ \text{s.t.}\ & \tfrac{1}{d} \textstyle\sum_{i=1}^{d} \alpha_i A_i \preceq \lambda I \\ & \textstyle\prod_i \alpha_i \geq 1,\ \alpha_i > 0 \end{aligned} \qquad \text{(OPTSDP)}$$

Semidefinite Relaxation

$$\max_{\|x\|=1} \prod_{i=1}^{d} \langle x, A_i x\rangle^{1/d} \qquad (\textsc{Opt})$$

rank-1 relax ↓ use AM-GM ↓

$$\begin{aligned}\max_{X} \;& \prod_{i=1}^{d} \langle X, A_i\rangle^{1/d} \\ \text{s.t.} \;& \operatorname{Tr}(X) = 1,\ X \succeq 0\end{aligned} \quad \overset{\text{dual}}{\longleftrightarrow} \quad \begin{aligned}\min_{\lambda,\alpha} \;& \lambda \\ \text{s.t.} \;& \frac{1}{d}\sum_{i=1}^{d} \alpha_i A_i \preceq \lambda I \\ & \prod_i \alpha_i \geq 1,\ \alpha_i > 0\end{aligned} \qquad (\textsc{OptSDP})$$

Theorem
Let $r = \operatorname{rank}(X) \leq n$, γ *be Euler's constant,* ϕ *be digamma function*

$$c_r(\mathbb{K})\, \textsc{OptSDP} \leq \textsc{Opt} \leq \textsc{OptSDP}$$

Semidefinite Relaxation

$$\max_{\|x\|=1} \prod_{i=1}^{d} \langle x, A_i x\rangle^{1/d} \qquad (\textsc{Opt})$$

rank-1 relax ↓   use AM-GM ↓

$$\begin{aligned} \max_{X}\ & \prod_{i=1}^{d} \langle X, A_i\rangle^{1/d} \\ \text{s.t.}\ & \operatorname{Tr}(X) = 1,\ X \succeq 0 \end{aligned} \quad \overset{\text{dual}}{\longleftrightarrow} \quad \begin{aligned} \min_{\lambda,\alpha}\ & \lambda \\ \text{s.t.}\ & \tfrac{1}{d}\sum_{i=1}^{d} \alpha_i A_i \preceq \lambda I \\ & \prod_i \alpha_i \geq 1,\ \alpha_i > 0 \end{aligned} \qquad (\textsc{OptSDP})$$

Theorem
Let $r = \operatorname{rank}(X) \leq n$, γ *be Euler's constant,* ϕ *be digamma function*

$$c_r(\mathbb{K})\ \textsc{OptSDP} \leq \textsc{Opt} \leq \textsc{OptSDP}$$

$$c_r(\mathbb{K}) = \begin{cases} \exp(-\gamma - \log 2 - \phi\left(\frac{r}{2}\right) + \log\left(\frac{r}{2}\right)) > 0.2807 & \textit{if } \mathbb{K} = \mathbb{R} \\ \exp(-\gamma - \phi(r) + \log(r)) > 0.5614 & \textit{if } \mathbb{K} = \mathbb{C} \end{cases}$$

Higher-order Relaxations

$\textsc{OptSDP}$ constructed using AM/GM inequality. If $\prod_i \alpha_i = 1$,

$$\prod_{i=1}^{d} \langle x, A_i x \rangle^{1/d} \leq x^T \left(\frac{1}{d} \sum_{i=1}^{d} \alpha_i A_i \right) x$$

Can we get a better bound with a higher-degree polynomial on the RHS?

Higher-order Relaxations

$\textsc{OptSDP}$ constructed using AM/GM inequality. If $\prod_i \alpha_i = 1$,

$$\prod_{i=1}^{d} \langle x, A_i x \rangle^{1/d} \leq x^T \left(\frac{1}{d} \sum_{i=1}^{d} \alpha_i A_i \right) x$$

Can we get a better bound with a higher-degree polynomial on the RHS?

Let E_k be elementary symmetric polynomials:

$$E_k(y_1, \ldots, y_d) = \binom{d}{k}^{-1} \sum_{I \subseteq [d], |I| = k} \prod_{i \in I} x_i$$

Higher-order Relaxations

$\textsc{OptSDP}$ constructed using AM/GM inequality. If $\prod_i \alpha_i = 1$,

$$\prod_{i=1}^{d} \langle x, A_i x\rangle^{1/d} \leq x^T \left(\frac{1}{d} \sum_{i=1}^{d} \alpha_i A_i \right) x$$

Can we get a better bound with a higher-degree polynomial on the RHS?

Let E_k be elementary symmetric polynomials:

$$E_k(y_1, \ldots, y_d) = \binom{d}{k}^{-1} \sum_{I \subseteq [d], |I| = k} \prod_{i \in I} x_i$$

Maclaurin's inequality:

$$(y_1 \cdots y_d)^{1/d} = E_d^{1/d} \leq E_{d-1}^{1/(d-1)} \leq \cdots \leq E_2^{1/2} \leq E_1 = \frac{y_1 + \cdots + y_d}{d}$$

Higher-order Relaxations

$\textsc{OptSDP}$ constructed using AM/GM inequality. If $\prod_i \alpha_i = 1$,

$$\prod_{i=1}^{d} \langle x, A_i x \rangle^{1/d} \le x^T \left(\frac{1}{d} \sum_{i=1}^{d} \alpha_i A_i \right) x$$

Can we get a better bound with a higher-degree polynomial on the RHS?

Let E_k be elementary symmetric polynomials:

$$E_k(y_1, \ldots, y_d) = \binom{d}{k}^{-1} \sum_{I \subseteq [d], |I| = k} \prod_{i \in I} x_i$$

Maclaurin's inequality:

$$(y_1 \cdots y_d)^{1/d} = E_d^{1/d} \le E_{d-1}^{1/(d-1)} \le \cdots \le E_2^{1/2} \le E_1 = \frac{y_1 + \cdots + y_d}{d}$$

Use Sum-of-Squares to construct relaxations $\textsc{OptSoS}_k$ for $1 \le k \le d$

$$\textsc{Opt} \le \textsc{OptSDP} = \textsc{OptSoS}_1 \le \textsc{OptSoS}_d$$

Trades off computation for accuracy

Example: Icosahedral form

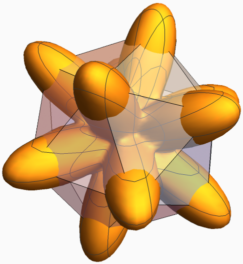
Let ψ be golden ratio, C chosen so that $\max_{x^2+y^2+z^2=1} p(x,y,z) = 1$.

$$p(x,y,z) = C\left[(x+\psi y)(x-\psi y)(y+\psi z)(y-\psi z)(z+\psi x)(z-\psi x)\right]^2$$

Example: Icosahedral form

Let ψ be golden ratio, C chosen so that $\max_{x^2+y^2+z^2=1} p(x,y,z) = 1$.

$$p(x,y,z) = C\left[(x+\psi y)(x-\psi y)(y+\psi z)(y-\psi z)(z+\psi x)(z-\psi x)\right]^2$$

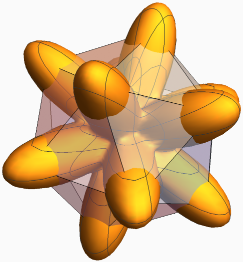


Hard instance because of high degree of symmetry!

Example: Icosahedral form

Let ψ be golden ratio, C chosen so that $\max_{x^2+y^2+z^2=1} p(x,y,z) = 1$.

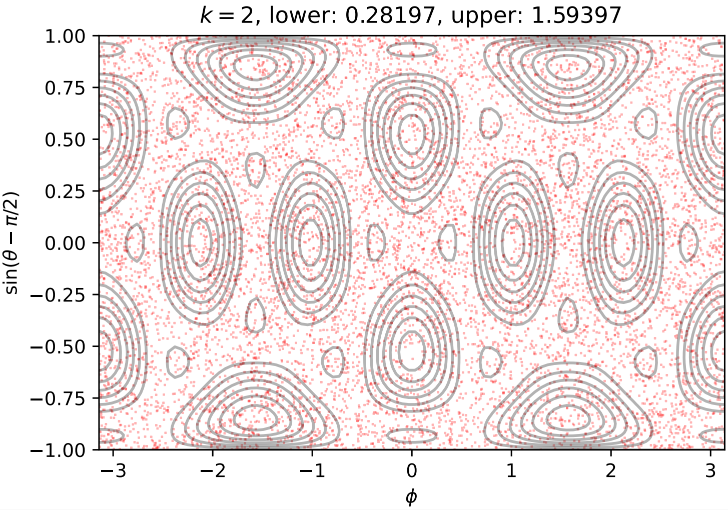
$$p(x,y,z) = C\left[(x+\psi y)(x-\psi y)(y+\psi z)(y-\psi z)(z+\psi x)(z-\psi x)\right]^2$$



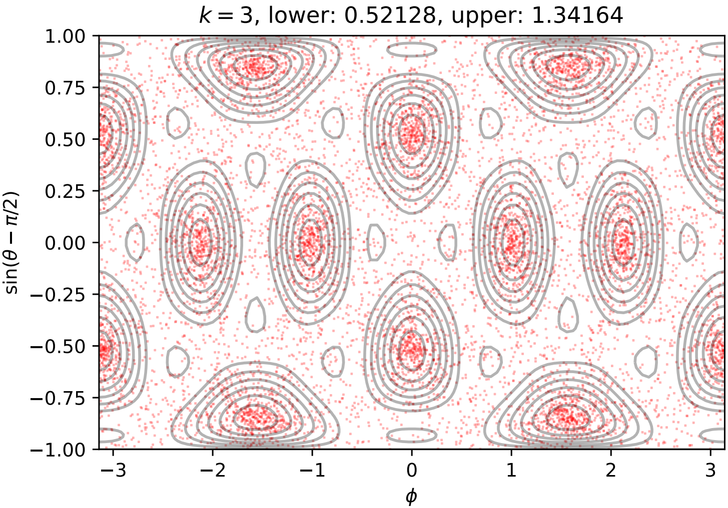
Hard instance because of high degree of symmetry!

We implement a randomized rounding algorithm to obtain feasible solution from relaxations OptSoS_k

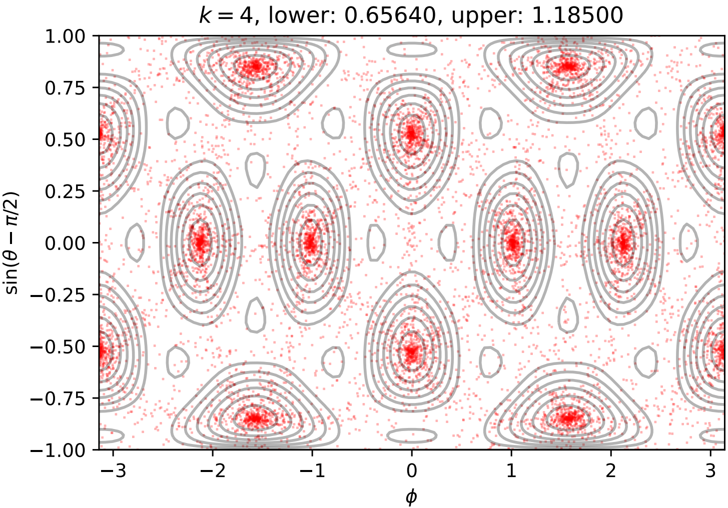
Distribution sampled from rounding algorithm



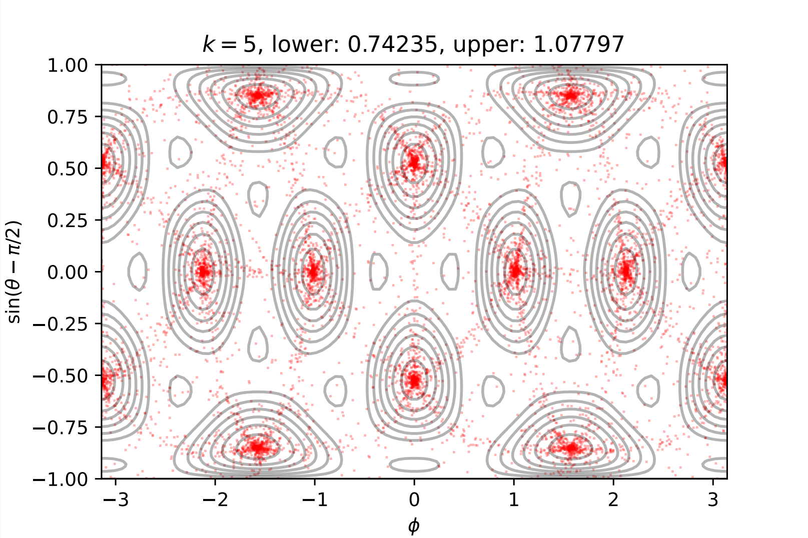
Distribution sampled from rounding algorithm



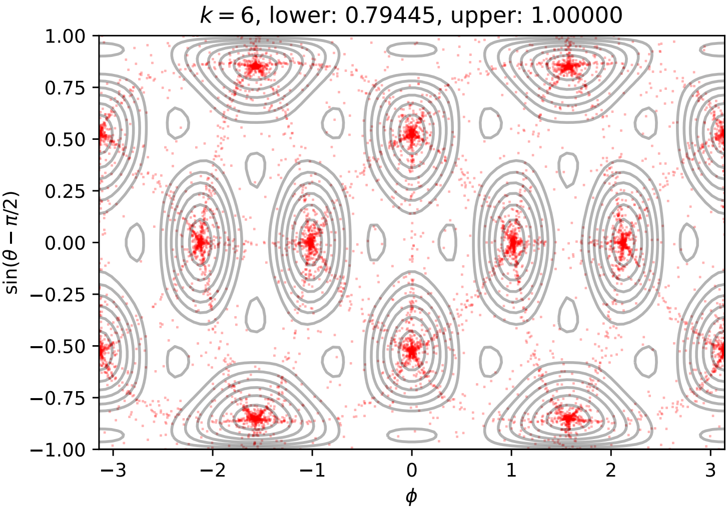
Distribution sampled from rounding algorithm



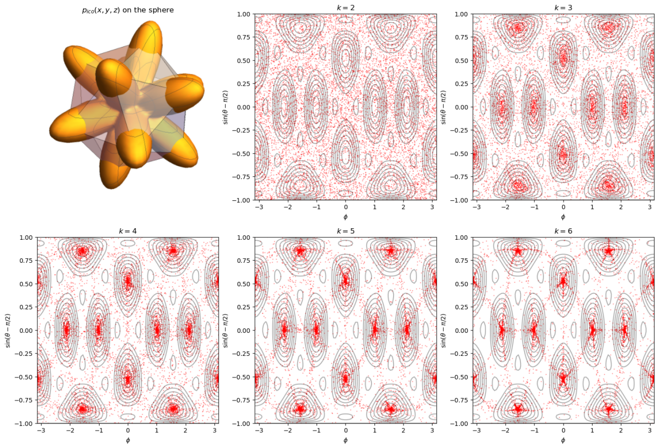
Distribution sampled from rounding algorithm



Distribution sampled from rounding algorithm



Distribution sampled from rounding algorithm



Distribution concentrates towards optima as k increases

Conclusion

Takeaways:

- Product of PSD forms has nice structure and generalizes many problems
- Exploiting product structure allows us to write a computationally efficient relaxation with good approximation guarantees

Conclusion

Takeaways:

- Product of PSD forms has nice structure and generalizes many problems
- Exploiting product structure allows us to write a computationally efficient relaxation with good approximation guarantees

Future work:

- Low-rank guarantees of solution from symmetry
- How to generate intermediate Sum-of-Squares relaxations for other high degree polynomial optimization problems?